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Drag on eccentrically positioned spheres translating and rotating in tubes

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The steady motion of an eccentrically positioned sphere in a circular cylindrical tube filled with viscous fluid is considered as a regular perturbation of the axisymmetric problem. A sequence of boundary-value problems is formulated involving Stokes equations and some linear boundary conditions. Solutions of the first- and second-order problems yield the leading terms in the perturbation series of the additional drag and the torque on the spheres. The results are found to be in good agreement with the previous off-axis solutions.

1. Introduction

A number of previous studies have considered the steady slow off-axis motion of a sphere in a fluid-filled tube. Happel and Brenner (1965) presented asymptotic solutions valid in the limit where the particle radius is small compared with the tube radius. They found that the drag on an eccentrically positioned sphere translating slowly in the axial direction is a function of the eccentricity parameter ϵ, which is defined as the ratio of the distance between the sphere centre and the tube axis to the tube radius. For constant values of particle radius and velocity the drag on the particle decreases as ϵ increases, attains a minimum for $\epsilon \approx 0{\cdot}4$ and then increases for greater values of ϵ. The solution by Happel & Brenner is based on the method of reflections, which requires that the particle be small and far removed from the tube walls. Solutions for the opposite limit of closely fitting particles are given by Bungay & Brenner (1973) in the form of singular perturbation series. Their results are in qualitative agreement with the results of Happel & Brenner.

Recently Tözeren (1982) considered the off-axis motion of a sphere translating in a tube as a regular perturbation of the axisymmetrical problem. His solutions based on the boundary collocation procedure described in Leichtberg, Pfeffer & Weinbaum (1976) are valid for a wide range of particle-to-tube-diameter ratios under the condition that ϵ is small. The zeroth-order perturbation solution yields the drag on a concentrically positioned sphere. This result was previously obtained by Haberman & Sayre (1958) and Leichtberg *et al.* (1976). The first-order solution gives the leading term in the perturbation expansion of the torque on the particle but yields no information about the drag. In the present paper the first correction to the zeroth-order drag is obtained by determining the second-order perturbation solutions. The present results are in good agreement with the results of Happel & Brenner (1965) when the radius of the sphere and ϵ are small. The results show that the drag on a finite sphere slightly off-axis is smaller than the drag on identical spheres flowing with the same velocity along the centreline.

The determination of the first few perturbation solutions, as ϵ tends to zero, is much simpler than tackling the more general problem involving arbitrary eccentricity. The

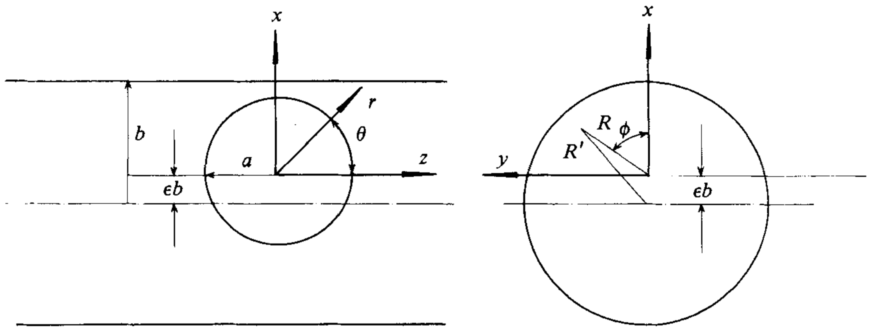

FIGURE 1. The flow of an eccentrically positioned sphere in a tube.

solution of Stokes equations in general involves surface spherical harmonics of any order (see Happel & Brenner 1965). However, the perturbation scheme given in this paper considerably simplifies the series solutions: the zeroth-order velocities are axisymmetric, first-order velocities are proportional to $\cos\phi$ (where ϕ is the polar angle), and second-order velocities that lead to non-zero drag are also axisymmetrical. This reduces substantially the amount of computational work.

The perturbation scheme and analytical solutions are presented in §2, 3 and 4. The results are discussed and compared with previous work in §5.

2. Formulation

Consider the slow translation and rotation of an eccentrically positioned sphere in a fluid-filled circular cylindrical tube (figure 1). The particle translates parallel to the axis with velocity U, rotates about the $+y$-axis with angular velocity $\epsilon\Omega$, and the viscous fluid flows with an average velocity $\frac{1}{2}V$. The sphere radius a is assumed to be comparable to the tube radius b. The distance between the sphere centre and the tube axis is taken as ϵb, where ϵ is small compared with unity.

The inertial terms are neglected in the Navier–Stokes equations, and the motion of the suspending fluid is considered to be a steady Stokes flow relative to the particles. The equations of Stokes flow are

$$\mu\nabla^2\mathbf{v} = \nabla p, \tag{2.1}$$

where $\mathbf{v}$ is the velocity vector and p is the pressure. The equation of continuity is

$$\nabla . \mathbf{v} = 0. \tag{2.2}$$

The velocities and pressure are treated as a regular perturbation of the axisymmetric problem

$$\mathbf{v} = \sum_{n=0}^{\infty} \mathbf{v}^{(n)}\epsilon^n, \quad p = \sum_{n=0}^{\infty} p^{(n)}\epsilon^n, \tag{2.3}$$

where each field $\mathbf{v}^{(n)}$ and $p^{(n)}$ satisfies the Stokes equations and equation of continuity

$$\mu\nabla^2\mathbf{v}^{(n)} = \nabla p^{(n)}, \quad \nabla . \mathbf{v}^{(n)} = 0. \tag{2.4}$$

Taking the sphere centre as the origin, spherical and cylindrical coordinate systems are introduced as shown in figure 1. The equation of the cylinder surface written with respect to the latter coordinate system depends on the perturbation quantity ϵ:

$$R = b\{1-\epsilon\cos\phi-\tfrac{1}{4}\epsilon^2(1-\cos 2\phi)-\tfrac{1}{32}\epsilon^4(\tfrac{3}{2}-2\cos 2\phi+\tfrac{1}{2}\cos 4\phi)+O(\epsilon^5)\} \tag{2.5}$$

as $\epsilon \to 0$.

The boundary conditions are (figure 1)

$$\left.\begin{aligned} \mathbf{v} &= U\mathbf{k}+a\epsilon\Omega\mathbf{j}\times\mathbf{e}_r \quad (r=a),\\ \mathbf{v} &= V\left(1-\frac{R'^2}{b^2}\right)\mathbf{k} \quad (z=\pm\infty),\\ \mathbf{v} &= 0 \quad (R'=b), \end{aligned}\right\} \tag{2.6}$$

where (R', z', ϕ') are cylindrical coordinates whose symmetry axis coincides with the axis of the tube. The unit vectors $\mathbf{j}, \mathbf{k}, \mathbf{e}_r$ are referred to the (R, z, ϕ) coordinate system.

Substituting (2.3) and the relation between R' and R, $R'^2 = R^2+2Rb\epsilon\cos\phi+b^2\epsilon^2$, into (2.6), the boundary conditions on the sphere and at infinity are obtained for various perturbation fields:

(i) On the sphere $(r=a)$

$$\left.\begin{aligned} \mathbf{v}^{(0)} &= U\mathbf{k},\\ \mathbf{v}^{(1)} &= a\omega\mathbf{j}\times\mathbf{e}_r, \end{aligned}\right\} \tag{2.7}$$

and higher-order fields are identically zero;

(ii) at infinity $(z=\pm\infty)$

$$\left.\begin{aligned} \mathbf{v}^{(0)} &= V\left(1-\frac{R^2}{b^2}\right)\mathbf{k},\\ \mathbf{v}^{(1)} &= -\frac{2RV}{b}\cos\phi\mathbf{k},\\ \mathbf{v}^{(2)} &= -V\mathbf{k}, \end{aligned}\right\} \tag{2.8}$$

and higher-order velocities tend to zero at infinity.

The no-slip condition

$$\mathbf{v}^{(0)}+\epsilon\mathbf{v}^{(1)}+\ldots \approx 0 \tag{2.9}$$

at the surface of the cylinder, $R'=b$, can be reduced into a sequence of boundary conditions by expanding each velocity field in Taylor series about $R=b$:

$$\mathbf{v}^{(i)} = [\mathbf{v}^{(i)}]_{R=b}+\sum_{k=1}^{\infty}\frac{1}{k!}(R-b)^k\left[\frac{\partial^k v^{(i)}}{\partial R^k}\right]_{R=b}. \tag{2.10}$$

Substituting (2.5) into (2.10) and the resulting equation in (2.9) and collecting the terms of same order in ϵ, a sequence of boundary conditions at $R=b$ is obtained. These conditions are given below in terms of the radial distance R made dimensionless with respect to the tube radius b:

$$\left.\begin{aligned} &\mathbf{v}^{(0)} = 0, \quad \mathbf{v}^{(1)} = \cos\phi\frac{\partial\mathbf{v}^{(0)}}{\partial R},\\ &\mathbf{v}^{(2)} = \cos\phi\frac{\partial\mathbf{v}^{(1)}}{\partial R}-\tfrac{1}{2}\cos^2\phi\frac{\partial^2\mathbf{v}^{(0)}}{\partial R^2}+\tfrac{1}{2}(1-\cos^2\phi)\frac{\partial\mathbf{v}^{(0)}}{\partial R},\\ &\mathbf{v}^{(3)} = \cos\phi\frac{\partial\mathbf{v}^{(2)}}{\partial R}-\tfrac{1}{2}\cos^2\phi\frac{\partial^2\mathbf{v}^{(1)}}{\partial R^2}+\tfrac{1}{2}(1-\cos^2\phi)\frac{\partial\mathbf{v}(1)}{\partial R}\\ &\qquad +\tfrac{1}{6}\cos^3\phi\frac{\partial^3\mathbf{v}^{(0)}}{\partial R^3}-\tfrac{1}{2}\cos\phi(1-\cos^2\phi)\frac{\partial^2\mathbf{v}^{(0)}}{\partial R^2} \end{aligned}\right\} (R=1). \tag{2.11}$$

A summary of the zeroth- and first-order perturbation solutions obtained by Tözeren (1982*a*) is given in §3. In §4, the second-order perturbation solution $\mathbf{v}^{(2)}$ is developed using these results.

3. Zeroth- and first-order perturbation solutions

The zeroth-order velocities in the present perturbation solution is the solution for the axisymmetrical motion of a sphere along the axis of a circular cylinder. This is given by Leichtberg *et al.* (1976) as a superposition of solutions of Stokes equations in spherical coordinates $\tilde{\mathbf{v}}^{(0)}$ and a cylindrical coordinates $\hat{\mathbf{v}}^{(0)}$. To carry out the superposition of these solutions ($\mathbf{v}^{(0)} = \tilde{\mathbf{v}}^{(0)} + \hat{\mathbf{v}}^{(0)}$) all vector components are referred to the cylindrical coordinates even when the spherical coordinates are being used as the independent variables. The R- and z-components of $\tilde{\mathbf{v}}^{(0)}$ are (see Leichtberg *et al.* 1976)

$$\left.\begin{aligned} \tilde{v}_z^{(0)} &= \sum_{n=2}^{\infty} (C_n P_n(\mu)\, r^{-n-1} + D_n(P_n + 2F_n) r^{-n+1}), \\ \tilde{v}_R^{(0)} &= \sum_{n=2}^{\infty} \left(C_n \frac{(n+1)\,F_n(\mu)}{\sin\theta}\, r^{-n-1} + D_n \frac{(n+1)\,F_{n+1} - 2\mu F_n}{\sin\theta}\, r^{-n+1}\right), \end{aligned}\right\} \tag{3.1}$$

where $P_n(\mu)$ and $F_n(\mu)$ are Legendre and Gegenbauer polynomials of order n and $\mu = \cos\theta$. The coefficients C_n and D_n are determined by the boundary conditions on the surface of the sphere. Owing to the symmetry about the $z = 0$ surface and the properties of Legendre and Gegenbauer polynomials, C_n and D_n are equal to zero when n takes odd values.

The solution applicable to axisymmetric creeping-flow problems in infinite cylinders is given by Leichtberg *et al.* (1976). Using matrix notation, this can be written as:

$$\begin{pmatrix} \hat{v}_R \\ 0 \\ \hat{v}_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ V(1-R^2) \end{pmatrix} + \int_0^{\infty} \mathbf{S}(0, zt)\, \mathbf{I}(R, t) \begin{pmatrix} A(t) \\ 0 \\ B(t) \end{pmatrix} dt, \tag{3.2}$$

where all the variables having dimensions of length are non-dimensionalized by the tube radius b, and the matrices $\mathbf{S}(\phi, zt)$ and $\mathbf{I}(R, t)$ are

$$\mathbf{S}(\phi, zt) = \begin{bmatrix} \cos\phi \sin zt & 0 & 0 \\ 0 & \sin\phi \sin zt & 0 \\ 0 & 0 & \cos\phi \cos zt \end{bmatrix},$$

$$\mathbf{I}(R, t) = \begin{bmatrix} tI_1(Rt) & 0 & RtI_0(Rt) \\ 0 & 0 & 0 \\ tI_0(Rt) & 0 & RtI_1(Rt) + 2I_0(Rt) \end{bmatrix}.$$

The functions $I_0(t)$ and $I_1(t)$ used in these equations are zeroth-and first-order modified Bessel functions. The functions $A(t)$ and $B(t)$ given by equations (3.13) and (3.14) of Leichtberg *et al.* (1976) are chosen such that the no-slip condition $\mathbf{v}^{(0)} = \tilde{\mathbf{v}}^{(0)} + \hat{\mathbf{v}}^{(0)} = 0$ on the cylindrical tube is automatically satisfied for arbitrary values of C_n and D_n:

$$\begin{pmatrix} A(t) \\ 0 \\ B(t) \end{pmatrix} = -\frac{2}{\pi}\, \mathbf{I}(1, t)^{-1} \begin{pmatrix} F_R(t) \\ 0 \\ F_z(t) \end{pmatrix}. \tag{3.3}$$

The functions $F_R(t)$ and $F_z(t)$ are Fourier transforms of the series in (3.1) giving $\tilde{v}_R$ and $\tilde{v}_z$ at $R = 1$ (sine transform for $\tilde{v}_R$ and cosine for $\tilde{v}_z$):

$$\begin{pmatrix} \tilde{v}_R \\ 0 \\ \tilde{v}_z \end{pmatrix} = \frac{2}{\pi} \int_0^{\infty} \mathbf{S}(0, zt) \begin{pmatrix} F_R \\ 0 \\ F_z \end{pmatrix} dt \quad (R = 1). \tag{3.4}$$

The solution obtained by the superposition of (3.1) and (3.2) subject to (3.3) satisfies the no-slip condition at $R = 1$. The coefficients C_n and D_n are then determined by applying the boundary conditions at the surface of the spheres to this solution (see Leichtberg *et al.* 1976).

The first-order velocities subject to the boundary conditions (using dimensionless variables a, r, R and z)

$$\mathbf{v}^{(1)} = ab\Omega\mathbf{j}\times\mathbf{e}_r \quad (r = a), \tag{3.5}$$

$$\mathbf{v}^{(1)} = -2RV\cos\phi\mathbf{k} \quad (z = \mp\infty), \tag{3.6}$$

$$\mathbf{v}^{(1)} = \cos\phi\,\frac{\partial\mathbf{v}^{(0)}}{\partial R} \quad (R = 1) \tag{3.7}$$

are determined in Tözeren (1982*a*) by the superposition of the following solutions:

(*A*) the solution of Stokes equations in cylindrical coordinates satisfying the boundary condition at the cylindrical surface (3.7) only; and

(*B*) superposition of solutions in spherical and cylindrical coordinates satisfying conditions at the particle surface and giving zero velocities at the tube surface.

The boundary conditions (3.5)–(3.7) indicate that the velocities are given by the terms proportional to $\cos\phi$ (or $\sin\phi$) in the general solutions of Stokes equations in spherical coordinates (see equation (3-2.3) of Happel & Brenner 1965)

$$\begin{aligned}\tilde{v}_R^{(1)} = \cos\phi \sum_{n=1}^{\infty}\Big\{&a_n\frac{r^{-n-1}}{(2n+1)\sin\theta}[nP^1_{n+1}+(n+1)P^1_{n-1}]\\ &+\tfrac{1}{2}b_n[r^{-n-2}P^2_{n+1}-n(n+1)\,r^{-n-2}P_{n+1}]\\ &+\frac{1}{n(2n-1)}c_n[-\tfrac{1}{4}(n-2)\,r^{-n}(P^2_{n+1}-n(n+1)\,P_{n+1})+(n+1)\,r^{-n}P^1_n\sin\theta]\Big\},\end{aligned} \tag{3.8 a}$$

$$\begin{aligned}\tilde{v}_\phi^{(1)} = \sin\phi \sum_{n=1}^{\infty}\Big\{&a_n\Big[-nr^{-n-1}\sin\theta\,P^1_{n+1}-\frac{1}{2(2n+3)}r^{-n-1}\\ &\times(nP^2_{n+2}+(n+3)\,P^2_n-n(n+1)\,(n+2)\,P_{n+2}-n(n+1)^2\,P_n)\Big]\\ &-\frac{1}{\sin\theta}b_n r^{-n-2}P^1_n+\frac{n-2}{2n(2n-1)\sin\theta}c_n r^{-n}P^1_n\Big\},\end{aligned} \tag{3.8b}$$

$$\begin{aligned}\tilde{v}_z^{(1)} = \cos\phi \sum_{n=1}^{\infty}\Big\{&-a_n r^{-n-1}P^1_n-b_n nr^{-n-2}P^1_{n+1}\\ &+c_n\Big[\frac{n}{2(2n+1)}r^{-n}P^1_{n+1}+\frac{(n+1)^2}{n(2n-1)\,(2n+1)}r^{-n}P^1_{n-1}\Big]\Big\},\end{aligned} \tag{3.8c}$$

and in cylindrical coordinates

$$\begin{pmatrix}\hat{v}_R^{(1)}\\ \hat{v}_\phi^{(1)}\\ \hat{v}_z^{(1)}\end{pmatrix} = \int_0^\infty \mathbf{S}(\phi, zt)\;\mathbf{I}_3(R,t)\begin{pmatrix}\chi\\ \phi\\ \pi\end{pmatrix}dt, \tag{3.9}$$

where $\mathbf{I}_3(R,t)$ is (see Happel & Brenner 1965, equation 7-3.51)

$$\mathbf{I}_3(R,t) = \begin{bmatrix}\dfrac{I_1(Rt)}{Rt} & I_1'(Rt) & RtI_1''(Rt)\\ -I_1'(Rt) & -\dfrac{I_1(Rt)}{Rt} & \dfrac{I_1(Rt)}{Rt}-I_1'(Rt)\\ 0 & I_1(Rt) & I_1(Rt)+RtI_1'(Rt)\end{bmatrix}$$

The solution (A) mentioned above is obtained by choosing $\chi(t)$, $\phi(t)$, $\pi(t)$ as

$$\begin{pmatrix}\chi(t)\\ \phi(t)\\ \pi(t)\end{pmatrix} = \frac{2}{\pi}\,\mathbf{I}_3(1,t)^{-1}\begin{pmatrix}0\\ 0\\ F(t)\end{pmatrix}, \tag{3.10}$$

where

$$\begin{aligned}F(t) = \sum_{n=2}^{\infty} (-1)^{\frac{1}{2}n}\,\frac{1}{n!}\{&C_n[t^{n+1}K_0(t)+t^nK_1(t)]\\ &-D_n[(n^2-3n+3)t^{n-1}K_0(t)-(2n-3)t^nK_1(t)+(n-2)(n-3)t^{n-2}K_1(t)]\}\\ &+\tfrac{1}{2}\pi[t^2I_1(t)C(t)+(t^2I_0(t)+2tI_1(t))D(t)],\end{aligned} \tag{3.11}$$

as given by equation (2.13) of Tözeren (1982a). The velocities obtained by substituting (3.10) into (3.9) satisfy the boundary condition (3.7). The K_0 and K_1 are zeroth- and first-order Macdonald functions.

The solution (B) giving zero velocities at $R = 1$ independent of the values of a_n, b_n and c_n in (3.8) is developed by Tözeren (1982) choosing χ, ϕ, π as

$$\begin{pmatrix}\chi\\ \phi\\ \pi\end{pmatrix} = -\frac{2}{\pi}\,\mathbf{I}_3(1,t)^{-1}\begin{pmatrix}G_R(1,t)\\ G_\phi(1,t)\\ G_z(1,t)\end{pmatrix}, \tag{3.12}$$

where $G_R(R,t)$, $G_\phi(R,t)$, $G_z(R,t)$ are Fourier transforms of $\tilde{v}_R$, $\tilde{v}_\phi$, $\tilde{v}_z$, in (3.8) with respect to z for fixed R (sine transform for $\tilde{v}_R$ and $\tilde{v}_\phi$, cosine for $\tilde{v}_z$):

$$\begin{aligned}G_R(R,t) = \sum_{n=2}^{\infty}\frac{(-1)^{\frac{1}{2}n}}{(n-1)!}\Big\{&a_{n-1}\,\frac{n-1}{R}\,[-(n-2)t^{n-2}K_1(Rt)+t^{n-1}K_0(Rt)]\\ &-\tfrac{1}{2}b_n(t^{n+1}K_2(Rt)+t^{n+1}K_0(Rt))+\frac{c_n}{2n-1}\Big[\frac{(n-2)^2(n-1)}{4n}\,t^{n-1}K_2(Rt)\\ &+\tfrac{1}{2}(2n-1)Rt^nK_1(Rt)-\tfrac{1}{4}(n+1)(n-2)t^{n-1}K_0(Rt)\Big]\Big\}\cos\phi,\end{aligned} \tag{3.13a}$$

$$\begin{aligned}G_\phi(R,t) = \sum_{n=2}^{\infty}\frac{(-1)^{\frac{1}{2}n}}{(n-1)!}\Big\{&-\tfrac{1}{2}a_{n-1}(n-1)\,[(n-2)t^{n-1}K_2(Rt)+nt^{n-1}K_0(Rt)]\\ &-b_n\,\frac{t^n}{R}\,K_1(Rt)\;\frac{c_n(n-2)}{2n(2n-1)\,R}\,[-(n-2)(n-1)t^{n-2}K_1(Rt)\\ &+(2n-1)Rt^{n-1}K_0(Rt)]\Big\}\sin\phi,\end{aligned} \tag{3.13b}$$

$$\begin{aligned}G_z(R,t) = \sum_{n=2}^{\infty}\frac{(-1)^{\frac{1}{2}n}}{(n-1)!}\Big\{&-a_{n-1}(n-1)t^{n-1}K_1(Rt)\\ &+b_nt^{n+1}K_1(Rt)+\tfrac{1}{2}c_n\Big[\frac{(n-1)(n^2+2)}{n(2n-1)}\,t^{n-1}K_1(Rt)-Rt^nK_0(Rt)\Big]\Big\}\cos\phi.\end{aligned} \tag{3.13c}$$

Using (3.12), at $R = 1$, $\hat{\mathbf{v}}$ is

$$\begin{aligned}\begin{pmatrix}\hat{v}_R\\ \hat{v}_\phi\\ \hat{v}_z\end{pmatrix} &= -\frac{2}{\pi}\int_0^\infty \mathbf{S}(\phi,zt)\,\mathbf{I}_3(1,t)\,\mathbf{I}_3(1,t)^{-1}\begin{pmatrix}G_R\\ G_\phi\\ G_z\end{pmatrix}dt\\ &= -\frac{2}{\pi}\int_0^\infty \mathbf{S}(\phi,zt)\begin{pmatrix}G_R\\ G_\phi\\ G_z\end{pmatrix}dt = -\begin{pmatrix}\tilde{v}_R\\ \tilde{v}_\phi\\ \tilde{v}_z\end{pmatrix}\end{aligned} \tag{3.14}$$

Therefore the superposition of $\tilde{\mathbf{v}}$ and $\hat{\mathbf{v}}$ leads to zero velocities at $R = 1$ independent of the values of a_n, b_n, c_n. These coefficients were then determined in Tözeren (1982) by applying the boundary conditions on the surface of the sphere to the superposition of solutions (A) and (B).

4. Second-order perturbation solution

The velocities $\mathbf{v}^{(0)}$ are axisymmetric; the R- and z-components of $\mathbf{v}^{(1)}$ are proportional to $\cos\phi$, the ϕ-component is proportional to $\sin\phi$. The components of $\mathbf{v}^{(2)}$ as indicated by the boundary conditions (2.7)–(2.11), consist of terms proportional to $\cos 2\phi$ (or $\sin 2\phi$) and axisymmetric terms. Among these surface spherical harmonics of different type in the general solution of Stokes equations in spherical coordinates (Happel & Brenner 1965, equation 3-2.3) the one that contributes to the drag on the spheres is p_{-2}. This term is a combination of $r^{-2}P_1(\mu)$ and $r^{-2}P_1^1(\mu)\cos\phi$. Yielding velocities proportional to $\cos\phi$ (or $\sin\phi$), the second term violates the boundary conditions for $\mathbf{v}^{(2)}$, and is therefore excluded from the series solution. Also, according to this result, the spherical harmonic proportional to $\cos 2\phi$, $r^{-n-1}P_n^2(\mu)\cos 2\phi$, gives no drag on the sphere. To evaluate the second-order corrections to drag on the sphere, it suffices then to determine the axisymmetrical part of $\mathbf{v}^{(2)}$: This solution satisfies the following boundary conditions (see (2.7), (2.8) and in particular (2.11)):

$$\mathbf{v}^{(2)} = 0 \quad (r = a), \tag{4.1a}$$

$$\mathbf{v}^{(2)} = -V\mathbf{k} \quad (z = \pm\infty), \tag{4.1b}$$

$$\mathbf{v}^{(2)} = \frac{1}{2}\left(\frac{\partial \bar{\mathbf{v}}^{(1)}}{\partial R}\right) + \frac{1}{4}\left(\frac{\partial \mathbf{v}^{(0)}}{\partial R} - \frac{\partial^2 \mathbf{v}^{(0)}}{\partial R^2}\right)\bigg|_{R=1}, \tag{4.1c}$$

where $\bar{\mathbf{v}}^{(1)}$ is that part of $\mathbf{v}^{(1)}$ independent of angle ϕ (i.e. the expression multiplied by $\cos\phi$ or $\sin\phi$ in $\mathbf{v}^{(1)}$).

As the zeroth-order solution (see Leichtberg *et al.* 1976), the axisymmetrical part of $\mathbf{v}^{(2)}$ can be found as a superposition of solutions in spherical and cylindrical coordinates, (3.1) and (3.2). Axisymmetric solutions of Stokes equations $\mathbf{u}$ which satisfy the boundary conditions at $R = 1$ and at infinity, (4.1 c, b) are developed by choosing the functions $C(t)$ and $D(t)$ as follows. The left-hand side of (4.1 c) is written as a Fourier cosine (or sine) inversion integral in the variable z. Using (3.1)–(3.13), this is found as the superposition of the following expressions.

(i) The second term in (4.1 c) and the series in spherical harmonics in (3.1)

$$\frac{\partial \tilde{v}_z^{(0)}}{\partial R} - \frac{\partial^2 \tilde{v}_z^{(0)}}{\partial R^2}\bigg|_{R=1} = \left(\frac{\partial}{\partial R} - \frac{\partial}{\partial R^2}\right)\{C_n P_n r^{-n-1} + D_n(P_n + 2F_n) r^{-n+1}\}\bigg|_{R=1}$$
$$= \frac{2}{\pi}\int_0^\infty (C_n M_3^n(t) + D_n M_4^n(t))\cos zt\, dt \tag{4.2a}$$

and

$$\frac{\partial \tilde{v}_R^{(0)}}{\partial R} - \frac{\partial^2 \tilde{v}_R^{(0)}}{\partial R^2}\bigg|_{R=1} = \frac{2}{\pi}\int_0^\infty (C_n M_1^n(t) + D_n M_2^n(t))\sin zt\, dt, \tag{4.2b}$$

where the functions $M_1(t)$, $M_2(t)$, $M_3(t)$ and $M_4(t)$ are obtained using equations (A 3)–(A 6) of Leichtberg *et al.* (1976):

$$M_1^n(t) = \frac{(-1)^{\frac{1}{2}n}}{n!}\{2tK_0(t) + 3K_1(t) + t^2K_1(t)\}t^n, \tag{4.3a}$$

$$M_2^n(t) = -\frac{(-1)^{\frac{1}{2}n}}{n!}\bigg\{(2n^2 - 8n + 9)K_0(t) + (n-2)(n-3)tK_1$$
$$+ 3(n-2)(n-3)\frac{K_1(t)}{t} - (2n-3)t^2K_0\bigg\}t^{n-1}, \tag{4.3b}$$

$$M_3^n(t) = \frac{(-1)^{\frac{1}{2}n}}{n!}\{tK_0(t)+2K_1(t)\}\, t^{n+1}, \tag{4.3c}$$

$$M_4^n(t) = \frac{(-1)^{\frac{1}{2}n}}{n!}\{2n(n-1)\, K_1(t)+n(n-1)tK_0(t) \\ -(2n-3)t^2K_1(t)\}\, t^{n-1}, \quad n = 2,4,6,\ldots. \tag{4.3d}$$

(ii) The second term in (4.1c) and the solution in cylindrical coordinates (3.2)

$$\left(\frac{\partial}{\partial R}-\frac{\partial^2}{\partial R^2}\right)\begin{pmatrix}\hat{v}_R^{(0)}\\ 0\\ \hat{v}_z^{(0)}\end{pmatrix} = \left(\frac{\partial}{\partial R}-\frac{\partial^2}{\partial R^2}\right)\int_0^\infty \mathbf{S}(0,zt)\,\mathbf{I}(R,t)\begin{pmatrix}C_0(t)\\ 0\\ D_0(t)\end{pmatrix}dt$$

$$= \int_0^\infty \mathbf{S}(0,zt)\,\mathbf{J}(t)\begin{pmatrix}C_0(t)\\ 0\\ D_0(t)\end{pmatrix}dt \quad (R=1), \tag{4.4}$$

where the elements of the matrix $\mathbf{J}(t)$ are obtained by taking the relevant derivatives of $\mathbf{I}(R,\, t)$ given in (3.2):

$$J_{11}(t) = 2t^2I_0(t)-3tI_1(t)-t^3I_1(t), \tag{4.5a}$$

$$J_{12}(t) = t(1-t^2)\, I_0(t), \tag{4.5b}$$

$$J_{21}(t) = t^2(2I_1(t)-tI_0(t)), \tag{4.5c}$$

$$J_{22}(t) = -2t^2I_0(t)+tI_1(t)\,(4-t^2), \tag{4.5d}$$

(iii) The first term $(\partial\tilde{\bar{v}}^{(1)}/\partial R)|_{R=1}$ in (4.1c) and the solution in cylindrical coordinates (3.9)

$$\frac{\partial}{\partial R}\int_0^\infty \begin{bmatrix}\sin zt & 0 & 0\\ 0 & \sin zt & 0\\ 0 & 0 & \cos zt\end{bmatrix}\mathbf{I}_3(R,\, t)\begin{pmatrix}\chi(t)\\ \phi(t)\\ \pi(t)\end{pmatrix}dt\Bigg|_{R=1}$$

$$= \int_0^\infty \begin{bmatrix}\sin zt & 0 & 0\\ 0 & \sin zt & 0\\ 0 & 0 & \cos zt\end{bmatrix}\mathbf{H}(t)\begin{pmatrix}\chi(t)\\ \phi(t)\\ \pi(t)\end{pmatrix}dt, \tag{4.6}$$

where derivatives of the components of $\mathbf{I}_3(R,\, t)$ with respect to R are

$$\left.\begin{aligned} &H_{11}(t) = I_0-\frac{2I_1}{t}, \quad H_{12} = t\left(\left(1+\frac{2}{t^2}\right)I_1-\frac{I_0}{t}\right)\\ &H_{13}(t) = I_0(t^2+2)-I_1\left(\frac{4}{t}+t\right), \quad H_{31} = 0,\\ &H_{32}(t) = tI_0-I_1, \quad H_{33}(t) = tI_0+t^2I_1. \end{aligned}\right\} \tag{4.7}$$

(iv) The term $(\partial\tilde{\bar{v}}^{(1)}/\partial R)|_{R=1}$ written as a Fourier inversion integral in the variable z:

$$\frac{\partial}{\partial R}\tilde{\bar{v}}_R^{(1)}\Big|_{R=1} = \frac{2}{\pi}\int_0^\infty F_R(t)\sin zt\, dt, \quad \frac{\partial}{\partial R}\tilde{\bar{v}}_z^{(1)}\Big|_{R=1} = \frac{2}{\pi}\int_0^\infty F_z(t)\cos zt\, dt. \tag{4.8}$$

Equations (3.13) giving the Fourier transforms of $v_z^{(1)}(z)$ and $v_R^{(1)}(z)$ in z are differentiated with respect to R to obtain F_R and F_z:

$$F_R(t) = \sum_{n=2}^{\infty} \frac{(-1)^{\frac{1}{2}n}}{(n-1)!} \Big\{ a_{n-1}(n-1)\,[(n-2)\,tK_0(t) - t^2K_1(t) + 2(n-2)\,K_1(t)]\,t^{n-2} + \tfrac{1}{2}b_n(\tfrac{3}{2}K_1(t) + \tfrac{1}{2}K_3(t))\,t^{n+2} + \frac{c_n}{(2n-1)} \times \left[\frac{(n-2)^2(n-1)}{8n} K_3(t) - \tfrac{1}{2}(2n-1)\,tK_0(t) + \frac{3n^3-7n^2+4n-4}{8n} K_1(t)\right] t^n \Big\}, \quad (4.9a)$$

$$F_z(t) = \sum_{n=2}^{\infty} \frac{(-1)^{\frac{1}{2}n}}{(n-1)!} \Big\{ a_{n-1}(n-1)\,t^n\left(K_0 + \frac{K_1}{t}\right) - b_n\,t^{n+2}\left(K_0 + \frac{K_1}{t}\right) + \tfrac{1}{2}c_n\left[-\left(\frac{(n-1)(n^2+2)}{n(2n-1)} + 1\right) tK_0 + t^2K_1 - \frac{(n-1)(n^2+2)}{n(2n-1)} K_1(t)\right] t^{n-1} \Big\}, \quad (4.9b)$$

where $a_n = 0$ for n even and $b_n = c_n = 0$ for n odd owing to the symmetry of the solution.

Finally, $\mathbf{v}^{(2)}$ at $R = 1$ can be written as

$$\mathbf{v}^{(2)} = \frac{1}{2}\frac{\partial}{\partial R}\mathbf{v}^{(1)} + \frac{1}{4}\left(\frac{\partial}{\partial R} - \frac{\partial^2}{\partial R^2}\right)\mathbf{v}^{(0)}\bigg|_{R=1}$$

or

$$\begin{pmatrix} \hat{v}_R^{(2)} \\ 0 \\ \hat{v}_z^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -V \end{pmatrix} + \int_0^\infty \mathbf{S}(0, zt) \begin{pmatrix} G_R \\ 0 \\ G_z \end{pmatrix} dt, \quad (4.10)$$

where

$$G_R(t) = \frac{1}{2\pi} \sum_{n=2}^{\infty} (C_n M_1^{(n)}(t) + D_n M_2^{(n)}(t)) + \tfrac{1}{4}(J_{11}(t)\,C_0(t) + J_{12}(t)\,D_0(t)) + \tfrac{1}{2}(H_{11}(t)\,\chi(t) + H_{12}(t)\,\phi(t) + H_{13}(t)\,\pi(t)) + \frac{1}{\pi} F_R(t), \quad (4.11a)$$

$$G_z(t) = \frac{1}{2\pi} \sum_{n=2}^{\infty} (C_n M_3^{(n)}(t) + D_n M_4^{(n)}(t)) + \tfrac{1}{4}(J_{21}\,C_0(t) + J_{22}\,D_0(t)) + \tfrac{1}{2}(H_{21}(t)\,\chi(t) + H_{22}(t)\,\phi(t) + H_{23}(t)\,\pi(t)) + \frac{1}{\pi} F_z(t). \quad (4.11b)$$

The coefficients a_n, b_n, c_n, C_n, D_n and the functions $A(t)$, $B(t)$, $\chi(t)$, $\phi(t)$, $\pi(t)$ giving zeroth- and first-order perturbation solutions have previously been computed in Tözeren (1982*a*). The solution of the Stokes equations satisfying the boundary conditions (4.1*b*, *c*) is, using (3.2),

$$\begin{pmatrix} u_R \\ u_\phi \\ u_z \end{pmatrix} = \int_0^\infty \mathbf{S}(0, zt)\,\mathbf{I}(R, t) \begin{pmatrix} C_u \\ 0 \\ D_u \end{pmatrix} dt, \quad (4.12)$$

where from (4.10)

$$\begin{pmatrix} C_u(t) \\ 0 \\ D_u(t) \end{pmatrix} = \mathbf{I}(1, t)^{-1} \begin{pmatrix} G_R(t) \\ 0 \\ G_z(t) \end{pmatrix}.$$

The solution $\mathbf{u}$ satisfies the boundary conditions at the cylinder surface and at infinity, (4.1 b, c). But, yielding non-zero velocities at the particle surface, it violates the boundary condition (4.1 a). This condition imposed on $\mathbf{v}^{(2)}$ can be satisfied by finding $\mathbf{w}$, a solution of Stokes equations, subject to the following boundary conditions:

$$\left.\begin{aligned} \mathbf{w} &= 0 \quad (R = 1, \quad z = \pm\infty), \\ \mathbf{w} &= -\mathbf{u} \quad (r = a). \end{aligned}\right\} \tag{4.14}$$

Similarly to the zeroth-order perturbation solution $\mathbf{v}^{(0)}$, the $\mathbf{w}$ can be found as the superposition of solutions in spherical and cylindrical coordinates (3.1) and (3.2) subject to (3.3) (see Leichtberg *et al.* 1976). The kernel functions $A(t)$ and $B(t)$ in (3.2) are chosen such that the homogeneous boundary conditions at $R = 1$ are automatically satisfied for arbitrary values of the coefficients C_n and D_n in (3.1), and these coefficients are then determined by applying the boundary condition at $r = a$ in (4.14). The $\mathbf{v}^{(2)}$ is given by the superposition of $\mathbf{w}$ and $\mathbf{u}$.

5. Torque and drag on off-centre spheres

The zeroth-, first- and second-order perturbation solutions and numerical tests that are performed to determine the accuracy of these solutions are discussed in this section.

The zeroth- and first-order solutions are given in Tözeren (1982 a). The zeroth-order solution was previously determined by Leichtberg *et al.* (1976) as a special case of the motion of a coaxial array of concentrically positioned spheres. Tözeren (1982 b) considered the same problem as an application of the boundary integral-equation method. Results of the previous work on spherical particles are compared with the present results in table 1. The variables used in comparisons are the coefficients of additional drag $\lambda^{(U)}$, $\lambda^{(V)}$ and $\lambda^{(\Omega)}$

$$F = 6\pi\mu a(\lambda^{(U)}U + \lambda^{(U)}V + \lambda^{(\Omega)}a\Omega), \tag{5.1}$$

where F is the total drag on the spheres. For a sphere flowing slightly off-centre the values in table 1 give the zeroth-order terms in the perturbation expansions of $\lambda^{(U)}$ and $\lambda^{(V)}$ as ϵ approaches zero:

$$\left.\begin{aligned} \lambda^{(U)} &= \lambda_0^{(U)} + \epsilon\lambda_1^{(U)} + \epsilon^2\lambda_2^{(U)} + \dots, \\ \lambda^{(V)} &= \lambda_0^{(V)} + \epsilon\lambda_1^{(V)} + \epsilon^2\lambda_2^{(V)} + \dots, \\ \lambda^{(\Omega)} &= \epsilon\lambda_1^{(\Omega)} + \epsilon^2\lambda_2^{(\Omega)} + \dots \end{aligned}\right\} \quad (\epsilon \to 0). \tag{5.2}$$

The results of Wang & Skalak (1969) given in table 1 are for an infinite chain of spheres uniformly distributed along the axis with spacing equal to 40 particle radii. The factor that influences the accuracy most is the numerical integration procedure used to compute the Fourier inversion integrals detailed in the previous sections. Extensive numerical tests were carried out by Tözeren (1982 b) to determine how the accuracy is influenced by the parameters of the Simpson's rule used in numerical integrations. These parameters are N = number of integration points and T = upper limit of integration replacing infinity in Fourier integrals. The N and T that lead to the smallest errors in these experiments are used in the numerical results presented in this section (see Tözeren 1982 b). Table 1 shows good agreement between the results of Tözeren (1982 b), Wang & Skalak (1969) and the present work. The discrepancies which are less than 0·1 % for $a/b \leqslant 0{\cdot}6$ are due to the type of polynomial approximation used for Bessel functions, Simpson's rule of integration, and the uniform spacing of

a/b	Leichtberg *et al.*		Present results		Wang *et al.*		Tözeren	
	$\lambda_0^{(U)}$	$\lambda_0^{(V)}$	$\lambda_0^{(U)}$	$\lambda_0^{(V)}$	$\lambda_0^{(U)}$	$\lambda_0^{(V)}$	$\lambda_0^{(U)}$	$\lambda_0^{(V)}$
0·1	1·263	1·255	1·263	1·255	1·263	1·255	1·263	1·255
0·2	1·680	1·636	1·680	1·635	1·680	1·635	1·680	1·635
0·3	2·373	2·231	2·371	2·229	2·370	2·229	2·370	2·229
0·4	3·599	3·223	3·593	3·216	3·592	3·216	3·591	3·216
0·5	5·973	5·017	5·952	4·999	5·949	4·996	5·947	4·995
0·6	11·20	8·696	11·11	8·627	11·10	8·617	11·09	8·611
0·7	25·29	17·91	24·77	17·54	24·70	17·49	24·66	17·46

TABLE 1. Comparison of present results ($\lambda_0^{(U)}$ and $\lambda_0^{(V)}$) with results of Leichtberg *et al.* (1976), Wang & Skalak (1969), and Tözeren (1982*b*)

collocation points used. Somewhat larger differences for $a/b = 0{\cdot}7$, also observed by Leichtberg *et al.* (1976, p. 159), are probably due to spacing the collocation points uniformly. Table 1 shows that the small differences between the results of Leichtberg *et al.* (1976) and others for $a/b = 0{\cdot}3$ becomes very significant at larger values of a/b. (The discrepancy is 1% for $a/b = 0{\cdot}6$ and 3% for $a/b = 0{\cdot}7$.) Especially, the disagreement between Leichtberg *et al.* (1976) and the present work is unexpected because both use the same numerical procedure: a boundary method based on solutions of Stokes equations, series in spherical coordinates and integral transforms in cylindrical coordinates. Studying interactions of an infinite chain of spheres, Wang & Skalak (1969) used series in spherical and cylindrical coordinates. The work of Tözeren (1982*b*) is based on the boundary integral-equation method. The disagreement between Leichtberg *et al.* (1976) and others is more obvious in the case of two-sphere solutions, as discussed by Tözeren (1982*b*). The results we obtained by an analysis similar to that of Leichtberg *et al.* (1976) support the results of Tözeren (1982*b*). Moreover, in two-sphere solutions presented by Leichtberg *et al.* (1976), the $\lambda^{(V)}$ is very little affected by variations in particle spacing, especially for larger values of a/b. This is in contradiction with the results of Wang & Skalak (1969).

In the series solution of Stokes equations in spherical coordinates as given by Happel & Brenner (1965), the only term that contributes to the drag on the particle is the solid spherical harmonic p_{-2}. In the first-order solution $\mathbf{v}^{(1)}$, which is proportional to $\cos\phi$ (or $\sin\phi$) the p_{-2} is necessarily of the form $c_1 P_1^1(\mu)\cos\phi$. However, the v_z derived from such a term is odd and v_R and v_ϕ are even functions of z. Such velocities may not satisfy the boundary conditions on the particle, and therefore c_1, the coefficient of p_{-2}, must be zero. $\lambda_1^{(U)}$, $\lambda_1^{(V)}$ and $\lambda_1^{(\Omega)}$ are also equal to zero for this reason.

The term that contributes to the torque on the spheres is $\chi_{-2} = \alpha_1 r^{-2} P_1^1(\mu)\sin\phi$. As opposed to p_{-2}, this term is present in the first-order solution (3.8). Tözeren (1982) calculated the coefficient a_1 for three cases: (i) $U \neq 0$, $V = \Omega = 0$; (ii) $V \neq 0$, $U = \Omega = 0$; (iii) $\Omega \neq 0$, $U = V = 0$. These coefficients $A_1^{(U)}$, $A_1^{(V)}$ and $A_1^{(\Omega)}$ are tabulated in table 2. The torque $\mathbf{T}$ in general cases can be computed using

$$\mathbf{T} = 8\pi\mu\alpha\mathbf{j}(A_1^{(U)}U + A_1^{(V)}V + A_1^{(\Omega)}\Omega a)\epsilon + O(\epsilon^3). \tag{5.3}$$

The coefficients of each term in the series for $\mathbf{v}^{(0)}$ and $\mathbf{v}^{(1)}$ are determined by a boundary method based on matching the boundary conditions exactly at a number of points on the spherical surface. In this method the number of terms in the truncated series (and thus the number of unknown coefficients a_n, b_n, c_n, C_n, D_n) is equal to the number of boundary points. Use of 13 uniformly distributed boundary points is shown

a/b	$A^{(U)}$	$A^{(V)}$	$A^{(\Omega)}$
0·001	$-1{\cdot}299\times10^{-6}$	$1{\cdot}0013\times10^{-3}$	$-1{\cdot}000$
0·01	$-1{\cdot}324\times10^{-4}$	$1{\cdot}0132\times10^{-2}$	$-1{\cdot}000001$
0·1	$-1{\cdot}615\times10^{-2}$	$1{\cdot}16\times10^{-1}$	$-1{\cdot}00074$
0·2	$-0{\cdot}8258\times10^{-1}$	$2{\cdot}805\times10^{-1}$	$-1{\cdot}00591$
0·3	$-2{\cdot}466\times10^{-1}$	$5{\cdot}332\times10^{-1}$	$-1{\cdot}0204$
0·4	$-6{\cdot}137\times10^{-1}$	$9{\cdot}544\times10^{-1}$	$-1{\cdot}0503$
0·5	$-1{\cdot}443$	$1{\cdot}727$	$-1{\cdot}1048$
0·6	$-3{\cdot}500$	$3{\cdot}353$	$-1{\cdot}200$
0·7	$-9{\cdot}569$	$7{\cdot}422$	$-1{\cdot}371$

TABLE 2. The coefficients $A^{(U)}$, $A^{(V)}$, $A^{(\Omega)}$ for several different values of diameter ratio a/b

$m \backslash n$	$\lambda = 0{\cdot}3$		$\lambda = 0{\cdot}6$	
	9	13	13	17
13	1·2259	1·2259	429·47	—
17	1·2259	1·2259	40·57	40·57
25	—	1·2259	40·55	40·55
49	—	—	—	40·55

TABLE 3. Convergence of second-order perturbation solutions for $a/b = 0{\cdot}3$ and $0{\cdot}6$, where m = number of boundary points and n = number of terms in the truncated series

by Tözeren (1982*a*) to yield convergence of 5 digits in C_n, D_n and four digits in a_n, b_n, c_n. However, for the determination of the coefficients of $\mathbf{v}^{(2)}$, this method with the choice of 13 boundary points led to very poor results, as shown in table 3. Considerable improvement is obtained by increasing the number m of boundary points but keeping the number of terms n in the truncated series constant and minimizing the error in the sense of least squares. Table 3 gives the results of various tests performed for $a/b = 0{\cdot}3$ and $0{\cdot}6$. This table shows that converged results are obtained using $n = m = 13$ for $a/b = 0{\cdot}3$ and using $n = 13$, $m = 25$ for $a/b = 0{\cdot}6$. A point-matching procedure using 13 boundary points is employed for $a/b \leqslant 0{\cdot}3$, and a boundary method with least-square error is used for $a/b > 0{\cdot}3$ using 13 terms in the series (actually only 7 owing to symmetry with respect to z) but minimizing the error at 25 boundary points.

The values of $\lambda_2^{(U)}$, $\lambda_2^{(V)}$ and $\lambda_2^{(\Omega)}$ are given in table 4 for $0{\cdot}001 \leqslant a/b \leqslant 0{\cdot}6$. (The difference between tables 3 and 4 for $a/b = 0{\cdot}6$ is due to the difference in the number of integration points of the numerical integration scheme.) Minus signs in the first two columns indicate that there is a decrease of drag as the sphere moves slightly away from the cylinder axis. The third column indicates that an off-centre sphere rotating about the $+y$-axis experiences a drag in the $-\mathbf{k}$-direction. These results compare well with the results of Happel & Brenner (1965) in the limit where a/b is small, and with Bungay & Brenner (1973) when a/b approaches unity. Happel & Brenner give the following expression for an off-axis sphere:

$$\left.\begin{aligned} \mathbf{F} &= 6\pi\mu a\,\mathbf{k}\,\left(V(1-\epsilon^2)-U\right)\left(1+f(\epsilon)\frac{a}{b}+\dots\right),\\ f(\epsilon) &= 2{\cdot}10444-0{\cdot}6977\epsilon^2+O(\epsilon^4). \end{aligned}\right\} \tag{5.4}$$

a/b	$\lambda_2^{(U)}$	$\lambda_2^{(V)}$	$\lambda_2^{(\Omega)}$
0·001	−0·0007009	−1·00281	$1·732 \times 10^{-6}$
0·01	−0·007281	−1·02879	$1·765 \times 10^{-4}$
0·1	−0·1266	−1·388	$2·152 \times 10^{-2}$
0·2	−0·5923	−2·235	0·1101
0·3	−1·226	−3·493	0·3288
0·4	−3·619	−6·717	0·8182
0·5	−11·38	−15·12	1·925
0·6	−40·62	−41·72	4·640
0·7	−191·9	−157·6	—

TABLE 4. Second-order coefficients of drag for various values of a/b

For $a/b = 0{\cdot}001$, using these formulae, $\lambda_2^{(U)}$ and $\lambda_2^{(V)}$ are found to be 0·0006977 and 1·00280 respectively, compared with 0·0007009 and 1·00281 in table 4. The disagreement in the third digit is because of the omission of terms of order $(a/b)^2$ in (5.4) (equation (7-3.96) of Happel & Brenner 1965).

Two major concerns of this and the earlier paper (Tözeren 1982*a*) were: (i) to determine the torque on eccentrically positioned spheres flowing in tubes; (ii) to show that the drag on a finite sphere slightly off-axis is smaller than the drag on an identical sphere translating with the same velocity along the centreline. This effect was previously observed by Happel & Brenner (1965) for the limiting case when the particle radius is very small compared with tube radius, by Bungay & Brenner (1973) for closely fitting spheres, and by some experimental investigators (see Happel & Brenner 1965). A possible reason for this decrease is that some effective distance between the particle and the boundaries is greater for slightly off-centre spheres. Another physical variable of some interest is the pressure drop in the tube due to the motion of the particles. The second-order terms in the asymptotic expansion of pressure drop in ϵ (first corrections to zeroth-order terms) may be determined by an analysis similar to that given in this paper. Happel & Brenner and Bungay & Brenner show that the pressure drop for eccentrically positioned spheres decreases like the drag.

The torque and drag on an off-axis sphere translating and rotating in a cylindrical tube are determined in the limit where the eccentricity is small. It is found that the drag decreases as the sphere moves away from the cylinder axis, and that this decrease is more significant as the diameter ratio increases. An eccentrically positioned sphere subject to no torque, flowing in an otherwise quiescent fluid, rotates about the $-\mathbf{j}$ axis. The Poiseuille flow past a stationary sphere makes the sphere rotate in the opposite direction. The formulae for drag and torque presented in this paper are expected to give reasonable results for moderate values of the eccentricity parameter because the neglected term in the coefficient of drag λ is $O(\epsilon^4)$ and $O(\epsilon^3)$ in the torque coefficient A.

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